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ABSTRACT

It is extremely difficult to extend in a natural way the ccncepts developed in high school geometry to motivate the concepts of trigonometry. To solve this difficultry, the author defines "sensed angles" as ordered pairs of rays and makes appropriate definitions which yield the familiar angle properties of plane geometry. He then shows how this definition also yields the familiar results of trigonometry. (Author/CT)

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# The Trigonometry of Sensed Angles — An Analogue to the Circular Functions

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## 1. Preliminary Remarks

In the standard developments of elementary Euclidean geometry, the concept of angle addition is usually concerned with the addition of measures of angles in a very restricted way. For example, in Elementary Geometry from an Advanced Standpoint by E. E. Moise (Addison-Wesley Publishing Company, 1963) the formal reference to angle addition is in the "Angle-Addition Postulate", which states:

(*) If D is in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

A completely similar definition appears in most, if not all, recent high school geometry texts.

The definition (*) rules out the possibility of adding two non-acute angles. In fact, the angle-measure function that is introduced in these treatments cuts out the possibility (short of introducing new definitions) of assigning negative measures to certain angles. In other words, it is clear from the definitions that the angles being treated are not directed angles. This being the case, it is extremely difficult to extend in a natural way the concepts developed in school geometry to get at the concepts of trigonometry. Taking the position that there is considerable merit in developing the concepts of elementary mathematics in a unified fashion, rather than as isolated topics of algegra, geometry, and trigonometry, it should be reasonable to seek out a definition for the sum of

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two angles which will enable one to move from the concepts of geometry into a development of the concepts of trigonometry.

Since the standard developments of Euclidean geometry do not contain appropriate suggestions for such a treatement, it is natural to turn to "higher level" sources for some hints as to how to proceed. There are, perhaps, two criteria to look for in these sources which seem to be suggested by the above discussion. First, the discussions sought after should pertain to oriented angles. Second, the discussions shall pertain to adding angles rather than to adding the measures of angles. None who have concerned themselves with unifying the concepts of elementary mathematics have taken approaches which are flexible enough to "bridge the gaps" between the developments of algebra, geometry, and trigonometry. [If someone has done this, it does not seem to be published yet.] So, in searching the literature for suggestions, the best that can be hoped for are suggestions as to reasonable approaches towards this end.

In this regard, Projective Geometry, Volume II, by O. Veblen and J. W. Young (Ginn and Company, 1918) contains a brief discussion on the assignment of measures to oriented angles. It is noted in this discussion that "whenever the measure of an angle is $\beta$, it is also $2k\pi + \beta$, where $k$ is any positive or negative integer. This indetermination can be removed by requiring that the measure $\beta$ chosen for any angle shall always satisfy a condition of the form $-\pi < \beta \leq \pi$." It is then brought out that if one wishes to assign measures to all angles in a consistent way, one must establish that the measure of the sum of two angles differs from the sum of the measures of these angles by $2k\pi$, for some integer k, where the sum of two angles is defined as follows:

(**) If a, b, c are any three rays (not necessarily distinct) with the same vertex, any angle $\angle a_1 c_1$ congruent to $\angle ac$ is said to be the sum of any two angles $\angle a_2 b_2$ and $\angle b_3 c_3$ such that $\angle a_2 b_2$ is congruent to $\angle ab$ and $\angle b_3 c_3$ is congruent to $\angle bc$. The sum $\angle a_1 c_1$ is denoted by $\angle a_2 b_2 + \angle b_3 c_3$.

A schematic drawing [Figures (a) and (b)] lays bare what is being proposed here for the sum of two angles:

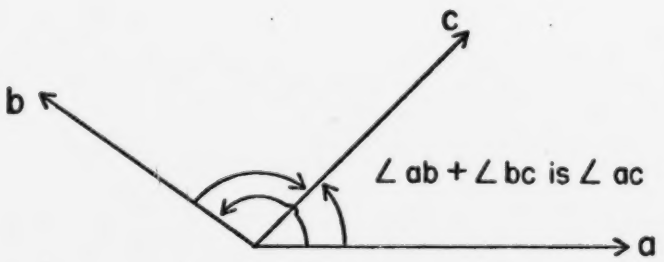


Figure (a)

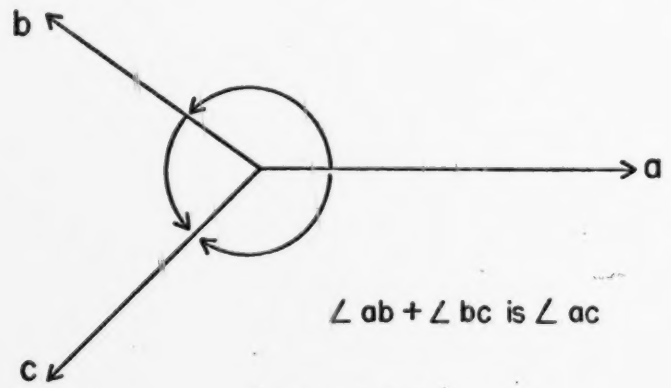


Figure (b)

The discussion is terminated at this point with the remark that "the trigonometric functions can now be defined, following the elementary textbooks, as the ratios of certain distances multiplied by ±1 according to appropriate conventions. This we shall take for granted in the future as having been carried out." It is evident that the authors, Veblen and Young, saw that the above-stated definition of angle sum leads naturally into a "standard" treatment of trigonometry.

A translation from the Russian, Geometric Transformations, Volume I, by P. S. Modenov and A. S. Parkhomenko (Academic Press, New York and London, 1965) suggests the inadequacy of the concept of "ordinary angle" in a discussion about formalizing the notion of rotation. For their purposes, the authors formalize the concept of an oriented angle — an ordered pair of rays — in which "∠MON is different from ∠NOM". In striving for an adequate definition for the sum of two oriented angles, they point out that a little thought

should convince one that if $\angle AOC = \alpha$ and $\angle B'O'D' = \beta$ [that is, if the measures of $\angle AOC$ and $\angle B'O'D'$ are $\alpha$ and $\beta$, respectively] then their sum should have a measure $\alpha + \beta$. Taking note of the fact that $\alpha + \beta$ might be greater than $2\pi$, for they allow the measures of angles to range from 0 to $2\pi$, the authors suggest that addition modulo $2\pi$ be used in computing measures of angle sums.

The definition of the sum of two oriented angles, as adopted by Modenov and Parkhomenko, is the following:

(Φ) Given oriented angles $\angle MON$ and $\angle N'O'P'$, the sum $\angle MON + \angle N'O'P'$ is the oriented angle $\angle MOP$, where $\angle NOP$ is the sensed angle with initial ray ON and congruent to $\angle N'O'P'$.

Comparing the treatments of angle sum suggested in (**) and (Φ), it is clear that both pairs of authors have handled the problem in a completely similar fashion.

As it will be demonstrated, a revised, but equivalent, form of these definitions can be used to generate "sensed angle equivalents" of the usual trigometric relations. This work is a natural extension of the concepts developed in an experimental geometry course under development at the UICSM and described in the writer's article "A Vector Approach to Euclidean Geometry" (The Mathematics Teacher, March, 1966).

## 2. The Trigonometry of Sensed Angles

A sensed angle is an ordered pair of rays. Given a ray r, we shall agree that $\vec{u}_r$ is the unit vector in the sense of r. In this regard, then, for sensed angle (a, b) we have by definition that

$$\cos(a, b) = \vec{u}_a \cdot \vec{u}_b \text{ and } \sin^{\perp}(a, b) = \vec{u}_a{}^{\perp} \cdot \vec{u}_b.$$

For the remainder of this paper, we assume that all sensed angles are in a fixed oriented plane $\pi$. Consider the following definition:

> Definition 1. Given sensed angles $(a, b)$ and $(c, d)$ of an oriented plane, $(a, b) + (c, d) = (a, t)$, where $t$ is the ray such that $(b, t) \cong (c, d)$, and such that $(b, t)$ and $(c, d)$ have the same sense.

Careful examination of this definition reveals that $(a, b) + (c, d)$ and $(c, d) + (a, b)$ do not yield the same sensed angle, for the former has initial ray $a$ while the latter has initial ray $c$. What the definition yields in this case is a pair of congruent sensed angles. Essentially, if the set of all sensed angles of a plane is partitioned into sets of congruent sensed angles, then Definition 1 can be interpreted as describing an "equivalence class" addition. That this addition is both commutative and associative will be demonstrated shortly.

Since the sensed angle $(b, a)$ is, in some respects, the "opposite" of sensed angle $(a, b)$, it is convenient to adopt the following definition for the difference of sensed angles:

> Definition 2. Given sensed angles $(a, b)$ and $(c, d)$, $(a, b) - (c, d) = (a, b) + (d, c)$.

Consider the orthonormal basis $(\vec{u}_b, \vec{u}_b^{\perp})$ for the given plane $\pi$, where $\vec{u}_b$ is the unit vector in the sense of a given ray $b$. Then, for each ray $x$ in $\pi$,

$$\text{(a)} \quad \vec{u}_x = \vec{u}_b(\vec{u}_b \cdot \vec{u}_x) + \vec{u}_b^{\perp}(\vec{u}_b^{\perp} \cdot \vec{u}_x)$$

and

$$\text{(b)} \quad \vec{u}_x^{\perp} = -\vec{u}_b(\vec{u}_b^{\perp} \cdot \vec{u}_x) + \vec{u}_b^{\perp}(\vec{u}_b \cdot \vec{u}_x)$$

Now, $\cos[(a, b) + (c, d)] = \cos(a, t)$, where $t$ is the ray specified in Definition 1. From (a) and the fact that $\cos(a, t) = \vec{u}_a \cdot \vec{u}_t$, we see that

$$\begin{aligned}
\cos(a,t) &= \vec{u}_a \cdot \vec{u}_t \\
&= [\vec{u}_b(\vec{u}_b \cdot \vec{u}_a) + \vec{u}_b^{\perp}(\vec{u}_b^{\perp} \cdot \vec{u}_a)] \cdot [\vec{u}_b(\vec{u}_b \cdot \vec{u}_t) \\
&\quad + \vec{u}_b^{\perp}(\vec{u}_b^{\perp} \cdot \vec{u}_t)] \\
&= (\vec{u}_b \cdot \vec{u}_a)(\vec{u}_b \cdot \vec{u}_t) + (\vec{u}_b^{\perp} \cdot \vec{u}_a)(\vec{u}_b^{\perp} \cdot \vec{u}_t) \\
&= (\vec{u}_a \cdot \vec{u}_b)(\vec{u}_b \cdot \vec{u}_t) - (\vec{u}_a^{\perp} \cdot \vec{u}_b)(\vec{u}_b^{\perp} \cdot \vec{u}_t) \\
&= \cos(a,b)\cos(b.t) - \sin^{\perp}(a,b)\sin^{\perp}(b,t) \\
&= \cos(a,b)\cos(c,d) - \sin^{\perp}(a,b)\sin^{\perp}(c,d)
\end{aligned}$$

This gives us:

Theorem 1. For sensed angles $(a,b)$ and $(c,d)$ of $\pi$, $\cos[(a,b) + (c,d)] = \cos(a,b)\cos(c,d) - \sin^{\perp}(a,b)\sin^{\perp}(c,d)$.

Since the objective of this development is to obtain "sensed angle equivalents" for the trigonometric functions, the result is quite encouraging. Notice that if $m(a,b) = x$ and $m(c,d) = y$ then Theorem 1 has a real number analogue $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

That the addition of sensed angles is commutative can now be demonstrated. To demonstrate that $(a,b) + (c,d)$ and $(c,d) + (a,b)$ are congruent (i.e., belong to the same equivalence class of congruent sensed angles) we note that sensed angles are congruent if and only if their cosines are the same. Since

$$\begin{aligned}
\cos[(a,b) + (c,d)] &= \cos(a,b)\cos(c,d) - \sin^{\perp}(a,b)\sin^{\perp}(c,d) \\
&= \cos(c,d)\cos(a,b) - \sin^{\perp}(c,d)\sin^{\perp}(a,b) \\
&= \cos[(c,d) + (a,b)]
\end{aligned}$$

it follows that $(a,b) + (c,d)$ and $(c,d) + (a,b)$ are congruent sensed angles. Hence, addition of sensed angles as defined is Definition 1 is commutative.

From Definition 2 and the facts that $\cos(d, c) = \cos(c, d)$ and $\sin^{\perp}(d, c) = -\sin^{\perp}(c, d)$, we make use of Theorem 1 to obtain:

Theorem 2. For sensed angles $(a, b)$ and $(c, d)$ of $\pi$,
$\cos[(a, b) - (c, d)] = \cos(a, b)\cos(c, d) + \sin^{\perp}(a, b)\sin^{\perp}(c, d)$.

Again we note that if $m(a, b) = x$ and $m(c, d) = y$ then the real number analogue of this theorem is the desired one, namely, $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

Making use of (b) and the fact that $\sin^{\perp}(a, t) = \vec{u}_a{}^{\perp} \cdot \vec{u}_t$, we see that $\sin^{\perp}[(a, b) + (c, d)] = \sin^{\perp}(a, t) = \vec{u}_a{}^{\perp} \cdot \vec{u}_t$, where $t$ is the ray specified in Definition 1. So,

$$\begin{aligned}
\sin^{\perp}[(a, b) + (c, d)] &= \vec{u}_a{}^{\perp} \cdot \vec{u}_t \\
&= [-\vec{u}_b(\vec{u}_b{}^{\perp} \cdot \vec{u}_a) + \vec{u}_b{}^{\perp}(\vec{u}_b \cdot \vec{u}_a)] \cdot [\vec{u}_b(\vec{u}_b \cdot \vec{u}_t) + \vec{u}_b{}^{\perp}(\vec{u}_b{}^{\perp} \cdot \vec{u}_t)] \\
&= -(\vec{u}_b{}^{\perp} \cdot \vec{u}_a)(\vec{u}_b \cdot \vec{u}_t) + (\vec{u}_b \cdot \vec{u}_a)(\vec{u}_b{}^{\perp} \cdot \vec{u}_t) \\
&= (\vec{u}_a{}^{\perp} \cdot \vec{u}_b)(\vec{u}_b \cdot \vec{u}_t) + (\vec{u}_a \cdot \vec{u}_b)(\vec{u}_b{}^{\perp} \cdot \vec{u}_t) \\
&= \sin^{\perp}(a, b)\cos(b, t) + \cos(a, b)\sin^{\perp}(b, t) \\
&= \sin^{\perp}(a, b)\cos(c, d) + \cos(a, b)\sin^{\perp}(c, d)
\end{aligned}$$

This gives us

Theorem 3. For sensed angles $(a, b)$ and $(c, d)$ of $\pi$,
$\sin^{\perp}[(a, b) + (c, d)] = \sin^{\perp}(a, b)\cos(c, d) + \cos(a, b)\sin^{\perp}(c, d)$.

From Definition 2, Theorem 3, and the facts that $\cos(c, d) = \cos(d, c)$ and $\sin^{\perp}(c, d) = -\sin^{\perp}(d, c)$, we obtain

Theorem 4. For sensed angles $(a, b)$ and $(c, d)$ of $\pi$,
$\sin^{\perp}[(a, b) - (c, d)] = \sin^{\perp}(a, b)\cos(c, d) - \cos(a, b)\sin^{\perp}(c, d)$.

For the record, we note that if $m(a, b) = x$ and $m(c, d) = y$ then the respective real number analogues of Theorems 3 and 4 are $\sin(x \pm y)$ $= \sin x \cos y \pm \cos x \sin y$.

That addition of sensed angles, as defined, is associative can now be demonstrated. This is done as follows:

$$\begin{aligned}
&\cos[\{(a,b) + (c,d)\} + (e,f)] \\
&= \cos\{(a,b) + (c,d)\}\cos(e,f) - \sin^{\perp}\{(a,b) + (c,d)\}\sin^{\perp}(e,f) \\
&= \{\cos(a,b)\cos(c,d) - \sin^{\perp}(a,b)\sin^{\perp}(c,d)\}\cos(e,f) \\
&\qquad -\{\sin^{\perp}(a,b)\cos(c,d) + \cos(a,b)\sin^{\perp}(c,d)\}\sin^{\perp}(e,f) \\
&= \cos(a,b)\cos(c,d)\cos(e,f) - \sin^{\perp}(a,b)\sin^{\perp}(c,d)\cos(e,f) \\
&\qquad - \sin^{\perp}(a,b)\cos(c,d)\sin^{\perp}(e,f) - \cos(a,b)\sin^{\perp}(c,d)\sin^{\perp}(e,f) \\
&= \cos(a,b)\{\cos(c,d)\cos(e,f) - \sin^{\perp}(c,d)\sin^{\perp}(e,f)\} \\
&\qquad - \sin^{\perp}(a,b)\{\sin^{\perp}(c,d)\cos(e,f) + \cos(c,d)\sin^{\perp}(e,f)\} \\
&= \cos(a,b)\cos\{(c,d) + (e,f)\} - \sin^{\perp}(a,b)\sin^{\perp}\{(c,d) + (e,f)\} \\
&= \cos[(a,b) + \{(c,d) + (e,f)\}]
\end{aligned}$$

It is not difficult to show each of the following properties of the sensed angle cos and $\sin^{\perp}$ functions:

1. $\cos(a,a) = 1$ and $\sin^{\perp}(a,a) = 0$
2. $\cos(a,a^{\perp}) = 0$ and $\sin^{\perp}(a,a^{\perp}) = 1$
3. $\cos(a,-a) = -1$ and $\sin^{\perp}(a,-a) = 0$
4. $\cos(a^{\perp},a) = 0$ and $\sin^{\perp}(a^{\perp},a) = -1$
5. $\cos(a,b) = \cos(b,a)$ and $\sin^{\perp}(a,b) = -\sin^{\perp}(b,a)$
6. $[\cos(a,b)]^2 + [\sin^{\perp}(a,b)]^2 = 1$
7. $\cos[(a,b) + (a,a^{\perp})] = -\sin^{\perp}(a,b)$
8. $\sin^{\perp}[(a,b) + (a,a^{\perp})] = \cos(a,b)$
9. $\cos[(a,b) + (a,-a)] = -\cos(a,b)$
10. $\sin^{\perp}[(a,b) + (a,-a)] = -\sin^{\perp}(a,b)$

11. $\cos[(a,b) + \{(a,-a) + (a,-a)\}] = \cos(a,b)$

12. $\sin^{\perp}[(a,b) + \{(a,-a) + (a,-a)\}] = \sin^{\perp}(a,b)$

For proofs of these properties, see the Appendix. The real number analogues of the above-mentioned properties are, respectively:

1′. $\cos 0 = 1$ and $\sin 0 = 0$

2′. $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$

3′. $\cos \pi = -1$ and $\sin \pi = 0$

4′. $\cos(-\frac{\pi}{2}) = 0$ and $\sin(-\frac{\pi}{2}) = -1$

5′. $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$

6′. $\cos^2 x + \sin^2 x = 1$

7′. $\cos(x + \frac{\pi}{2}) = -\sin x$

8′. $\sin(x + \frac{\pi}{2}) = \cos x$

9′. $\cos(x + \pi) = -\cos x$

10′. $\sin(x + \pi) = -\sin x$

11′. $\cos(x + 2\pi) = \cos x$

12′. $\sin(x + 2\pi) = \sin x$

The way is now clear to develop a complete analogue with sensed angles to the standard development of the circular functions. In short, the suggested trigonometry of sensed angles "bridges the gap" between the concepts of geometry and those of trigonometry. The theorems 1-4 and suggested properties 1-12 about the sensed angle functions cos and $\sin^{\perp}$ were given to "point the way."

It is interesting to note that in "Characterization of the Sine and Cosine" by H. E. Vaughan (The American Mathematical Monthly, December, 1955) it is proved that from Theorem 2 and the so-called limit law for sine, $\lim_{x \to 0} \frac{\sin x}{x} = 1$, all properties of sine and cosine may be derived. Since we can

prove Theorem 2 directly from Definition 2 (and vector properties), we need only to postulate the sc[illegible]ed angle form of the limit law for sine in order to successfully bridge the gap from geometry to trigonometry.

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# APPENDIX

1. $\cos(a,a) = \vec{u}_a \cdot \vec{u}_a = 1$, where $\vec{u}_a$ is the unit vector in the sense of ray a.
   $\sin^{\perp}(a,a) = \vec{u}_a{}^{\perp} \cdot \vec{u}_a = 0$, for $\vec{u}_a{}^{\perp}$ and $\vec{u}_a$ are orthogonal vectors.
   [In words, this says that the cos and $\sin^{\perp}$ of a null sensed angle are, respectively, 1 and 0.]

2. $\cos(a,a^{\perp}) = \vec{u}_a \cdot \vec{u}_a{}^{\perp} = 0$, for $\vec{u}_a$ and $\vec{u}_a{}^{\perp}$ are orthogonal vectors.
   $\sin^{\perp}(a,a^{\perp}) = \vec{u}_a{}^{\perp} \cdot \vec{u}_a{}^{\perp} = 1$.
   [In words, cos and $\sin^{\perp}$ of a right positively sensed angle are, respectively, 0 and 1.]

3. $\cos(a,-a) = \vec{u}_a \cdot \vec{u}_{-a} = \vec{u}_a \cdot -\vec{u}_a = -(\vec{u}_a \cdot \vec{u}_a) = -1$.
   $\sin^{\perp}(a,-a) = \vec{u}_a{}^{\perp} \cdot \vec{u}_{-a} = \vec{u}_a{}^{\perp} \cdot -\vec{u}_a = -(\vec{u}_a{}^{\perp} \cdot \vec{u}_a) = 0$.
   [In words, the cos and $\sin^{\perp}$ of a straight sensed angle are, respectively, $-1$ and 0.]

4. $\cos(a^{\perp},a) = \vec{u}_a{}^{\perp} \cdot \vec{u}_a = 0$. $\sin^{\perp}(a^{\perp},a) = \vec{u}_a{}^{\perp\perp} \cdot \vec{u}_a = -\vec{u}_a \cdot \vec{u}_a = -1$.
   [In words, the cos and $\sin^{\perp}$ of a negatively sensed right angle are, respectively, 0 and $-1$.]

5. $\cos(a,b) = \vec{u}_a \cdot \vec{u}_b = \vec{u}_b \cdot \vec{u}_a = \cos(b,a)$
   $\sin^{\perp}(a,b) = \vec{u}_a{}^{\perp} \cdot \vec{u}_b = -\vec{u}_a \cdot \vec{u}_b{}^{\perp} = -\vec{u}_b{}^{\perp} \cdot \vec{u}_a = -\sin^{\perp}(b,a)$.
   [In words, cos is an even function and $\sin^{\perp}$ is an odd function.]

6. From (a), $\vec{u}_b = \vec{u}_a(\vec{u}_a \cdot \vec{u}_b) + \vec{u}_a{}^{\perp}(\vec{u}_a{}^{\perp} \cdot \vec{u}_b)$. So, $\vec{u}_b \cdot \vec{u}_b = 1$ and $\vec{u}_b \cdot \vec{u}_b = (\vec{u}_a \cdot \vec{u}_b)^2 + (\vec{u}_a{}^{\perp} \cdot \vec{u}_b)^2$. Since $\cos(a,b) = \vec{u}_a \cdot \vec{u}_b$ and $\sin^{\perp}(a,b) = \vec{u}_a{}^{\perp} \cdot \vec{u}_b$, we have $[\cos(a,b)]^2 + [\sin^{\perp}(a,b)]^2 = 1$.

7. $\cos[(a,b) + (a,a^{\perp})] = \cos(a,b)\cos(a,a^{\perp}) - \sin^{\perp}(a,b)\sin^{\perp}(a,a^{\perp})$
   $= -\sin^{\perp}(a,b)$.

8. $\sin^{\perp}[(a,b) + (a,a^{\perp})] = \sin^{\perp}(a,b)\cos(a,a^{\perp}) + \cos(a,b)\sin^{\perp}(a,a^{\perp}) = \cos(a,b)$.

9. $\cos[(a,b) + (a,-a)] = \cos(a,b)\cos(a,-a) - \sin^{\perp}(a,b)\sin^{\perp}(a,-a)$
$= -\cos(a,b)$

10. $\sin^{\perp}[(a,b) + (a,-a)] = \sin^{\perp}(a,b)\cos(a,-a) + \cos(a,b)\sin^{\perp}(a,-a) = -\sin^{\perp}(a,b)$.

11. $\cos[(a,b) + \{(a,-a) + (a,-a)\}] = \cos(a,b)\cos\{(a,-a) + (a,-a)\} - \sin^{\perp}(a,b)\cdot$
$\sin^{\perp}\{(a,-a) + (a,-a)\} = \cos(a,b)\cos(a,a) - \sin^{\perp}(a,b)\sin^{\perp}(a,a) = \cos(a,b)$.
[In words, the period of cos is two straight angles.]

12. $\sin^{\perp}[(a,b) + \{(a,-a) + (a,-a)\}] = \sin^{\perp}(a,b)\cos\{(a,-a) + (a,-a)\} + \cos(a,b)\cdot$
$\sin^{\perp}\{(a,-a) + (a,-a)\} = \sin^{\perp}(a,b)\cos(a,a) + \cos(a,b)\sin^{\perp}(a,a) = \sin^{\perp}(a,b)$.
[In words, the period of $\sin^{\perp}$ is two straight angles.]